2-D vector rocket equations with air drag

1. Burnout time

The average mass of the rocket during boost is

(1.1) $m_A = m_D + \frac{m_P}{2}$

m_A = Average mass [kg]
m_D = Rocket dry mass [kg]
m_P = Propellant mass [kg]

The coast mass is equal to the rocket dry mass.

The air drag F_D for the rocket is given by

(1.2) $F_D = k \cdot v_{t_B}^2 = \frac{1}{2} \cdot \rho \cdot C_{D_R} \cdot A_R$

v_{t_B} = Burnout velocity [$\frac{m}{s}$]
ρ = Air density [$\frac{kg}{m^3}$]
C_{D_R} = Drag coefficient of the rocket [-] (0.75 for average rockets)
A_R = Rocket cross-sectional area [m^2]

The thrust of the rocket F_T is then

(1.3) $F_T = T - m_A \cdot g - k \cdot v_{t_B}^2$

T = Motor thrust [N]
g = Acceleration of gravity [$\frac{m}{s^2}$]

The definition of force is

(1.4) $F = m \cdot a$

As the mass is constant the factor rule in differentiation allows the mass to move outside the derivative operator, and the equation becomes

(1.5) $F = m \cdot \frac{dv}{dt}$

(1.6) $m_A \cdot \frac{dv}{dt} = T - m_A \cdot g - k \cdot v^2$

(1.7) $m_A \cdot \frac{dv}{dt} = T - m_A \cdot g - k \cdot v^2$

(1.8) $dt = \frac{m_A \cdot dv}{T - m_A \cdot g - k \cdot v^2} = \frac{m_A \cdot dv}{k \cdot \frac{T - m_A \cdot g}{k} - k \cdot v^2}$

Substituting $x^2 = \frac{T - m_A \cdot g}{k}$ yields to

(1.9) $dt = \frac{m_A \cdot dv}{k \cdot x^2 - k \cdot v^2} = \frac{m_A}{k} \cdot \frac{dv}{x^2 - v^2}$

To get now the burnout time t_B, we integrate over v_{t_B}

(1.10) $t_B = \frac{m_A}{k} \cdot \int \left(\frac{1}{x^2 - v^2} \right) dv_{t_B}$

(1.11) $t_B = \frac{m_A}{k} \cdot \frac{\ln(x + v_{t_B}) - \ln(v_{t_B} - x)}{2x} + C$

With the condition $t_B = 0$ (and thus $v_{t_B} = 0$) we can determine the integration constant C:

(1.12) $\frac{m_A}{k} \cdot \frac{\ln(x) - \ln(-x)}{2x} + C = 0$

(1.13) $C = \frac{m_A}{k} \cdot \frac{\ln(-x) - \ln(x)}{2x}$

(1.14)
$$t_B = \frac{m_A}{k} \cdot \frac{\ln(x + v_{t_B}) - \ln(v_{t_B} - x)}{2x} + \frac{m_A}{k} \cdot \frac{\ln(-x) - \ln(x)}{2x} = \frac{m_A}{2 \cdot k \cdot x} \cdot \left(\ln(x + v_{t_B}) - \ln(v_{t_B} - x) + \ln(-x) - \ln(x) \right)$$

(1.15)

$$t_B = \frac{m_A}{k} \cdot \frac{\ln(x + v_{t_B}) - \ln(v_{t_B} - x)}{2x} + \frac{m_A}{k} \cdot \frac{\ln(-x) - \ln(x)}{2x} = \frac{m_A}{2 \cdot k \cdot x} \cdot \left(\ln\left(\frac{x + v_{t_B}}{x} \right) + \ln\left(\frac{-x}{v_{t_B} - x} \right) \right)$$

(1.16) $$t_B = \frac{m_A}{2 \cdot k \cdot x} \cdot \ln\left(\frac{x + v_{t_B}}{x - v_{t_B}} \right)$$

Burnout time equation

2. Burnout velocity

Solving now for v_{t_B} :

(1.17) $$\frac{2 \cdot k \cdot x}{m_A} \cdot t_B = \ln\left(\frac{x + v_{t_B}}{x - v_{t_B}} \right)$$

Substitution $y = \frac{2 \cdot k \cdot x}{m_A}$

(1.18) $$y \cdot t_B = \ln\left(\frac{x + v_{t_B}}{x - v_{t_B}} \right)$$

(1.19) $$e^{y \cdot t_B} = \frac{x + v_{t_B}}{x - v_{t_B}}$$

(1.20) $$x \cdot e^{y \cdot t_B} - x = v_{t_B} \left(1 + \cdot e^{y \cdot t_B} \right)$$

(1.21) $$v_{t_B} = x \cdot \frac{e^{y \cdot t_B} - 1}{e^{y \cdot t_B} + 1}$$

Burnout velocity equation

The burnout time t_B can be also written as

$$t_B = \frac{I_{sp}}{T} = \frac{v_e}{g \cdot T}$$

I_{sp} = Specific impulse [s]

v_e = Effective exhaust velocity [$\frac{m}{s}$]

3. Burnout altitude

To get now the burnout velocity h_B we need to integrate 1.21 over the burnout time:

$$(1.22)\ h_B = \int \left(x \cdot \frac{e^{y \cdot t} - 1}{e^{y \cdot t} + 1} \right) dt_B = \frac{2 \cdot x \cdot \ln\left(e^{y \cdot t_B} + 1\right) - t_B \cdot x \cdot y}{y} + C$$

$$(1.23)\ h_B = \frac{2 \cdot x \cdot \ln\left(e^{y \cdot t_B} + 1\right) - t_B \cdot x \cdot y}{y} + C$$

With the condition $h_B = 0$ (and thus $t_B = 0$) we can determine the integration constant C:

$$(1.24)\ \frac{2 \cdot x \cdot \ln(2)}{y} + C = 0$$

$$(1.25)\ C = -\frac{2 \cdot x \cdot \ln(2)}{y}$$

$$(1.26)\ h_B = \frac{2 \cdot x}{y} \cdot \ln\left(\frac{e^{y \cdot t_B} + 1}{2}\right) - t_B \cdot x$$

Substitution $y = \frac{2 \cdot k \cdot x}{m_A}$

$$(1.27)\ h_B = \frac{m_A}{k} \cdot \ln\left(\frac{e^{y \cdot t_B} + 1}{2}\right) - t_B \cdot x$$

Burnout altitude equation 1

Replacing $e^{y \cdot t_B}$ by

$$(1.28)\ \frac{x + v_{t_B}}{x - v_{t_B}} = e^{y \cdot t_B}$$

yields to

$$(1.29)\quad h_B = \frac{m_A}{k} \cdot \ln\left(\frac{x}{x - v_{t_B}}\right) - t_B \cdot x$$

$$(1.30)\quad t_B = \frac{m_A}{2 \cdot k \cdot x} \cdot \ln\left(\frac{x + v_{t_B}}{x - v_{t_B}}\right)$$

$$(1.31)\quad h_B = \frac{m_A}{k} \cdot \ln\left(\frac{x}{x - v_{t_B}}\right) - \left(\frac{m_A}{2 \cdot k \cdot x} \cdot \ln\left(\frac{x + v_{t_B}}{x - v_{t_B}}\right)\right) \cdot x$$

$$(1.32)\quad h_B = \frac{m_A}{k} \cdot \ln\left(\frac{x}{x - v_{t_B}}\right) - \frac{m_A}{2 \cdot k} \cdot \ln\left(\frac{x + v_{t_B}}{x - v_{t_B}}\right)$$

$$(1.33)\quad h_B = \frac{2 \cdot m_A}{2 \cdot k} \cdot \ln\left(\frac{x}{x - v_{t_B}}\right) - \frac{m_A}{2 \cdot k} \cdot \ln\left(\frac{x + v_{t_B}}{x - v_{t_B}}\right)$$

$$(1.34)\quad h_B = \frac{m_A}{2 \cdot k} \cdot \ln\left(\frac{x}{x - v_{t_B}}\right)^2 - \frac{m_A}{2 \cdot k} \cdot \ln\left(\frac{x + v_{t_B}}{x - v_{t_B}}\right)$$

$$(1.35)\quad h_B = \frac{m_A}{2 \cdot k} \cdot \ln\left(\frac{x^2}{\left(x - v_{t_B}\right)^2} \cdot \frac{x - v_{t_B}}{x + v_{t_B}}\right)$$

$$\boxed{(1.36)\quad h_B = \frac{m_A}{2 \cdot k} \cdot \ln\left(\frac{x^2}{x^2 - v_{t_B}^2}\right)}$$

Burnout altitude equation 2

4. Coasting altitude

After the rocket has reached the burnout altitude, the so called coasting phase begins. The rocket has then the dry mass m_D and the initial velocity v_{t_B}. To derive the coast altitude h_C we start again with the definition of force:

$$(1.37)\quad F = m \cdot a = m \cdot v \cdot \frac{dv}{dh}$$

(1.38) $m_D \cdot v \cdot \frac{dv}{dh} = -m_D \cdot g - k \cdot v^2$

(1.39) $dh = \frac{m_D \cdot v \cdot dv}{-m_D \cdot g - k \cdot v^2}$

(1.40) $dh = \frac{m_D \cdot v \cdot dv}{k \cdot \frac{-m_D \cdot g}{k} - k \cdot v^2}$

Substitution $z^2 = \frac{-m_D \cdot g}{k}$

(1.41) $dh = \frac{m_D}{k} \cdot \frac{v \cdot dv}{z^2 - v^2}$

(1.42) $dh = \frac{m_D}{k} \cdot \frac{v \cdot dv}{z^2 - v^2}$

(1.43) $h_C = \frac{m_D}{k} \cdot \int \left(\frac{v}{z^2 - v^2} \right) dv_{t_B}$

(1.44) $h_C = \frac{m_D}{k} \cdot \int \left(\frac{v_{t_B}}{z^2 - v_{t_B}^2} \right) dv_{t_B} = \frac{m_D}{2 \cdot k} \cdot \ln\left(z^2 - v_{t_B}^2 \right) + C$

With the condition $h_C = 0$ (and thus $v_{t_B} = 0$) we can determine the integration constant C:

(1.45) $\frac{m_D}{2 \cdot k} \cdot \ln\left(z^2 \right) + C = 0$

(1.46) $C = -\frac{m_D}{2 \cdot k} \cdot \ln\left(z^2 \right)$

(1.47) $h_C = \frac{m_D}{2 \cdot k} \cdot \ln\left(z^2 - v_{t_B}^2 \right) - \frac{m_D}{2 \cdot k} \cdot \ln\left(z^2 \right)$

(1.48) $h_C = \frac{m_D}{2 \cdot k} \cdot \ln\left(\frac{z^2 - v_{t_B}^2}{z^2} \right)$ **Coast altitude equation**

5. Coast time

To determinate the time t_C from v_{t_B} to 0 (coasting time), the starting point is again the definition of force. The acceleration is here negative:

(1.49) $F = m \cdot (-a) = m \cdot \left(-\frac{dv}{dt}\right)$

(1.50) $m_C \cdot \left(-\frac{dv}{dt}\right) = -m_C \cdot g - k \cdot v^2$

(1.51) $dt = m_C \cdot \frac{dv}{m_C \cdot g + k \cdot v^2}$

(1.52) $dt = m_C \cdot \frac{dv}{k \cdot \frac{m_C \cdot g}{k} + k \cdot v^2}$

Substitution $z_a{}^2 = \frac{m_C \cdot g}{k}$

(1.53) $dt = \frac{m_C}{k} \cdot \frac{dv}{z_a^2 + v^2}$

(1.54) $t_C = \frac{m_C}{k} \cdot \int \left(\frac{1}{z_a^2 + v^2}\right) dv_{t_B} = \frac{m_C}{k} \cdot \frac{\arctan\left(\frac{v_{t_B}}{z_a}\right)}{z_a} + C$

With the condition $t_C = 0$ (and thus $v_{t_B} = 0$) we can determine the integration constant C:

(1.55) $\frac{m_C}{k} \cdot \frac{\arctan\left(\frac{0}{z_a}\right)}{z_a} + C = 0 \Rightarrow C = 0$

(1.56) $t_C = \frac{m_C}{k \cdot z_a} \cdot \arctan\left(\frac{v_{t_B}}{z_a}\right)$ **Coast time equation**

6. Parachute size

The desired maximum decent velocity for a rocket is 3-5 $\frac{m}{s}$. Therefore the parachute size needs to be accordingly calculated.

The maximum velocity v_E is then reached if the acceleration $a = 0$, hence

(1.57) $F = m \cdot a = 0 = m_D \cdot g - \frac{1}{2} \cdot C_{D_P} \cdot \rho \cdot A_P \cdot v^2$

(1.58) $v_E = \sqrt{\frac{2 \cdot m_D \cdot g}{C_{D_P} \cdot \rho \cdot A_P}}$

C_{D_P} = Drag coefficient of the rocket [-] (0.75 for a flat sheet used for a parachute, or 1.5 for a true dome-shaped chute).
A_R = Parachute area [m^2]

Solving for A_P:

(1.59) $A_P = \frac{2 \cdot m_D \cdot g}{v_E^2 \cdot C_{D_P} \cdot \rho}$

The chute area is $A_P = \frac{\pi \cdot D_P^2}{4}$, so the chute diameter is

(1.60) $D_P = \sqrt{\frac{8 \cdot m_D \cdot g}{\pi \cdot v_E^2 \cdot C_{D_P} \cdot \rho}}$ **Parachute diameter equation**